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Nonuniversal features of forced two-dimensional turbulence in the energy range

S. Danilov[1,*] and D. Gurarie[2,*]
[1]Institute of Atmospheric Physics, 109017 Moscow, Russia
[2]Mathematics Department, Case Western Reserve University, Cleveland, Ohio 44106-7044
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We examine energy spectra, fluxes, and transfers of two-dimensional forced incompressible turbulence with linear drag in the energy range, and find marked departures from 5/3 law and the idea of locality. Any attempt to bring the system into the ''ideal cascade state'' would result either in spectral (bulge) or flux distortion. We corroborate this observation by DNS (spectral code) and eddy-damped quasinormal Markovian simulations. We examine the energy peak wave number k_p, in terms of drag coefficient λ, and energy dissipation rate ε, and find a relation $k_p \sim C(\lambda^3/\varepsilon)^{1/2}$ to hold with $C \approx 50$, but only within a limited range of parameters.

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The standard phenomenology of two-dimensional (2D) turbulence calls for two inertial ranges, energy and enstrophy (based on its two conserved integrals), with the $k^{-5/3}$ spectrum in the energy range (below the source). Many authors have claimed to produce it ([1–7]), with a few exceptions [8], who questioned its validity and universality.

The upscale energy cascade to small k in 2D turbulence necessitates a proper dissipation mechanism to arrest energy accumulation at the gravest modes, and stabilize it. The natural choice is the linear (bottom) drag, combined with viscous dissipation at high k. So we could write the Fourier expansion of the vorticity equation for $\zeta = \Delta\psi = \Sigma \zeta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, as

$$\partial_t \zeta_{\mathbf{k}} + J_{\mathbf{k}} = -(\lambda + \nu k^{2n})\zeta_{\mathbf{k}} + f_{\mathbf{k}}, \tag{1}$$

where $J_{\mathbf{k}}$ denotes the Jacobian $J(\psi,\zeta) = \partial_x\psi\partial_y\zeta - \partial_y\psi\partial_x\zeta$, ν the viscosity coefficient (exponent $n=1$ corresponds to ordinary viscosity, while $n>1$ gives hyperviscosity), and f the source of vorticity. Clearly, uniform (scale-independent) drag λ, destroys ''dissipation free inertial interval,'' and leads to significant changes of the energy transfers and fluxes at low k.

So all previous attempts to reproduce the ''universal inverse cascade'' would either confine spectral range to a close proximity of the source, or impose some selective scale dependent dissipation at low k, or parametrize the ''large-scale backscatter'' onto inertial-range modes, pretending the system could be formally extended into smaller and smaller k.

We mention two examples of scale-selective friction: step-wise linear drag confined to the lowest modes [3], and hypo-friction $-\lambda k^{-2n}\zeta$, defined by negative powers of the Laplacian [8,9].

Step-wise linear drag created the desired effect ($k^{-5/3}$ energy spectra) in paper [3], but it had limited spatial resolution and moderate Reynolds numbers. Borue [8] performed high-resolution long-time simulations with hypo-friction, and his work led to unexpected results. While steep hypo-friction made spectral fluxes nearly uniform, the energy spectra deviated strongly from the classical $-5/3$ slope, up to -3. Recent paper [6] corroborated and explained these findings, by showing that any attempt to sharply arrest the inverse cascade would lead to large deviations.

Linear drag, unlike hypofriction (or stepwise friction), would arrest the inverse cascade smoothly, and thus produce spectra close to $k^{-5/3}$ over a sizable fraction of the energy range [7]. Yet, a closer examination, presented below, shows such spectra to be at odds with the conventional ideas of ''locality'' and ''constant flux.'' Indeed, varying forcing and dissipation allows some control of the spectral slope and the flux. However, the closer one brings the slope to $-5/3$, the more the spectral energy flux will deviate from a uniform distribution over the $k^{-5/3}$ interval, and vice versa; uniform flux results in a steepened slope. This result could explain, in particular, the discrepancy in the reported values of the Kolmogorov constant C_K in the 5/3 law (see, i.e., [3]).

The linear drag appears naturally in many physical models. For instance, it is used to model the effective drag created by turbulent planetary boundary layer on large-scale atmospheric flows. It tends to arrest the inverse cascade at some wavelength $2\pi/k_p$, associated with the energy peak. This leads to the basic (still largely open) problem of parametrizing it in terms of the external forcing-dissipation. We addressed the issues of nonuniversality and parametrization of k_p in a series of numeric experiments.

I. DNS EXPERIMENTS

We carried out numerical simulations by a fully dealiased pseudospectral method [10] at resolution 512^2. Time stepping was performed by the third order Adams-Bashforth method, and forcing was implemented like in [3]. Namely, we took a Markovian process: $f_{\mathbf{k},j+1} = A(1-r^2)^{1/2}e^{i\theta} + r f_{\mathbf{k},j}$ (j denotes time step), of amplitude A, correlation radius $1/(1-r)$, and random uniform phase-distribution θ on $[0,2\pi]$, localized within narrow spectral range (k_f-2, k_f+2) in the vicinity of forcing wave-number k_f. In most experiments we used two values: $k_f=100$ and 150. We also varied coefficient r, $r=0.9$ and 0.5, but found little effect.

We used different types of hyperviscosity: $n=2$ for the forcing scale $k_f=100$, and $n=8$ for $k_f=150$. The system was integrated to a stationary state, and then we computed

*Present address: NCAR, P.O. Box 3000, Boulder, CO 80307.

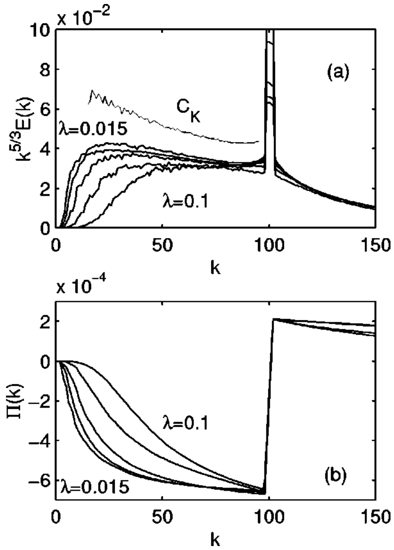

FIG. 1. Compensated energy spectra (a) and spectral energy fluxes (b) in DNS experiments for $\lambda=0.015$, 0.02, 0.03, 0.05, and 0.1 with $k_f=100$. Thin line in (a) shows the Kolmogorov constant C_K for $\lambda=0.03$.

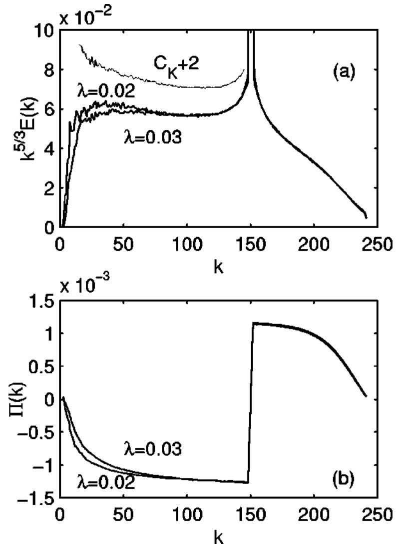

FIG. 2. Compensated energy spectra (a) and spectral energy fluxes (b) in DNS experiments for $\lambda=0.02$, and 0.03 with $k_f=150$. Thin line in (a) shows C_K+2 for $\lambda=0.03$.

stationary spectra by averaging over long time series.

Depending on λ the inverse cascade could penetrate to different levels (in terms of the energy peak wave-number k_p), before friction would stabilize it. In cases of low $k_p=5-10$, spectra of realizations show large fluctuations (though total energy remained stable). So long time averaging $O(1/\lambda)$, may not guarantee a smooth mean spectrum, particularly at small k. The spectra below are obtained by averaging over time intervals ranging from $1/\lambda$ to $2/\lambda$.

Figure 1(a) shows time-averaged compensated energy spectra $E(k)k^{5/3}$ from experiments with $k_f=100$ performed for various λ, but equal forcing. Figure 1(b) shows the corresponding energy fluxes $\Pi(k)$, defined as $\Pi(k)=-\mathrm{Re}\Sigma_0^k\psi_{\mathbf{k}}^*J_{\mathbf{k}}$ (the asterisk denotes complex conjugation and Re the real part). In those experiments λ varied from 0.1 to 0.015 through 0.05, 0.03 and 0.02. The compensated spectra are more or less flat over significant portion of the energy range, which indicates spectral slope close to $-5/3$. Small λ, however, leads to a buildup of spectral bulge at the lower end of the $-5/3$ interval. There exist an optimal λ that gives the least deviation from the $k^{-5/3}$ law and the smallest bulge. As λ drops below this value, the deviations from $k^{-5/3}$ grow stronger.

The spectral energy flux exhibits an opposite trend. Clearly, bottom drag would not sustain a uniform flux in the energy range. The surprising fact, however, comes in the relation between spectral slopes and fluxes. The most flat (ideal $k^{-5/3}$) spectrum for moderate $\lambda=0.05$ in Fig. 1(a), corresponds to a highly nonuniform flux. In general, the spectral plateau extends far beyond the regions of uniform flux. Conversely, decreasing λ flattens the energy flux, yet it builds a high spectral bulge at the lower end, so $E(k)$ deviates further from the $k^{-5/3}$ law.

We come to the conclusion that whatever mechanism sustains the $k^{-5/3}$ spectra in the energy range, it should have little bearing on the standard notion of ''constant flux'' and ''locality.'' The latter would yield $E(k)\sim|\Pi(k)|^{2/3}k^{5/3}$, ruled out by Figs. 1 and 2. Correspondingly, the Kolmogorov constant C_K, defined as $E(k)k^{5/3}|\Pi(k)|^{-2/3}$, changes from 4.5 near the source, to 7.2 close to the energy peak [thin line in Fig. 1(a)]. Since all standard estimates of C_K would try to fit it in the (undetermined) spectral interval, variations of the reported values are not surprising (see also [3]).

The appearance of spectral bulge also involves subtle nonlocal processes, which couple energy carrying modes $k\sim k_p$ to the forcing scale k_f (and above $k>k_f$). Our experiments show that a poorly resolved enstrophy range tends to suppress the enstrophy production, and at once drive the bulge down, so the compensated spectra would become flatter. Here we present an example (Fig. 2) of the averaged energy spectra and fluxes for $\lambda=0.02$ and 0.03 as obtained in simulations with $k_f=150$, i.e., enstrophy resolution $k_{\max}/k_f\approx1.6$ versus $k_{\max}/k_f\approx2.4$ in Fig. 1. Compared to Fig. 1, the energy spectra are much flatter [despite the greater value of $\Pi(k)$]. Notice, however, that they still exhibit the reverse ''flat energy-spectrum'' versus ''flat flux-spectrum'' relation, as in Fig. 1.

Similar relations could be observed in the high-resolution simulations by Boffetta *et al.* [7] (their Fig. 2), and the laboratory measurements by Paret and Tabeling [11] [compare their Figs. 2(b) and 3]. All these results clearly indicate variable energy flux, in regions of the sustained $-5/3$ energy slope. Both papers report an average value of C_K over a certain range of wave numbers, and ignore the variation of the quantity $E(k)k^{5/3}|\Pi(k)|^{-2/3}$.

The appearance of spectral bulge in our experiments could be associated with the formation of strong, long-lived vortices in the physical space, observed by Borue [8]. In the linear drag case these vortices are less robust than the hypofrictional ones [8], their size rarely exceeding the forcing scale by a factor of 2 or 3, with the vorticity level capped below 10 rms. We saw such vortices emerge from the local

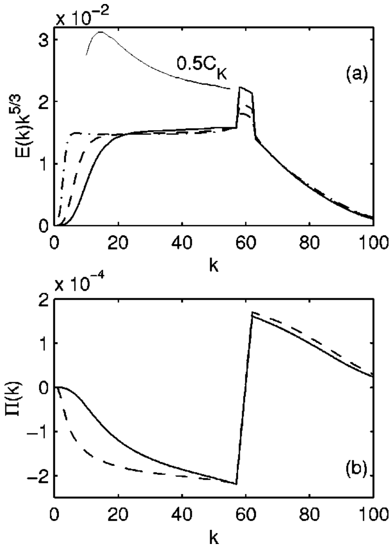

FIG. 3. Compensated energy spectra (a) and spectral energy fluxes (b) in EDQM simulations with linear drag with $\lambda=0.03$ (solid lines) and $\lambda=0.01$ (dashed lines). Thin line in (a) shows $0.5C_K$ for $\lambda=0.03$.

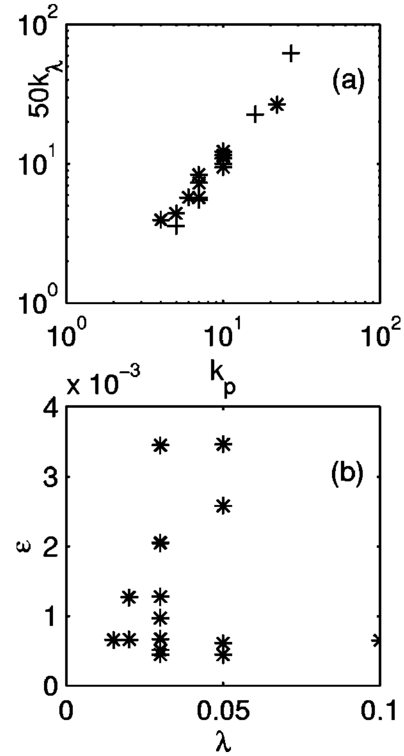

FIG. 4. (a) $50k_\lambda$ vs k_p for all DNS runs (*,+); (+) correspond to the series shown in Fig. 1. (b) Parameters λ,ε for all DNS runs.

extrema in the vorticity field, and found them in all realizations of the system, under different conditions. Their formation, however, was strongly suppressed in situations with poorly resolved enstrophy range, or sufficiently high total (viscous plus drag) dissipation at the forcing scale.

When the energy spectrum remained close to $k^{-5/3}$, the vortices seemed to be short lived and of little dynamic importance. Lowering drag coefficient λ made them more sparse and intense, though vortex size (with vorticity truncated above 2 rms) remained fairly stable, 2–3 times the source scale. The vorticity kurtosis deviated from the Gaussian value 3. However, even in simulations with relatively low $\lambda=0.015$, as in Fig. 1, it remained moderately high (around 7). Such a picture contrasts sharply with the hypofriction experiments, where kurtosis could grow higher order in magnitude. Intense vortices create strong circulation zones around them. While their size remains small relative to π/k_p, their main contribution to the energy comes from circulation zones, and those could extend well into the energy carrying (bulge) region. We discuss the ‘‘physical space structure’’ of 2D turbulence, and its effect on spectra and non-universality in a future publication [12].

II. EDQNM CLOSURE

The eddy-damped quasinormal Markovian (EDQNM) closure model of isotropic 2D turbulence is formulated for the energy spectrum $E(k)$. It parametrizes the transfer term T_k as a functional of $E(k)$, in terms of relaxation times θ_{kpq} for triads $\mathbf{k}=\mathbf{p}+\mathbf{q}$. The standard form $\theta_{kpq}=1/(\mu_k+\mu_p+\mu_q)$ (see [13], Chap. 8), involves combined shearing rate: $\mu_k=a[\int_0^k l^2E(l)\,dl]^{1/2}$ of large eddies ($l<k$) on the $\mathbf{k}$-mode, with coefficient $a=O(1)$.

EDQNM integration was performed at 128 resolution with forcing scale $58<k<62$, and forcing power, $\int F_k\,dk=0.0004$. EDQNM shares some features of the full system. For instance, if one allows an infinitely long enstrophy range the energy would go upscale, so its flux $|\Pi(k)|$ near forcing range would be equal to the forcing power. In our case the enstrophy range is poorly resolved, so the maximum of $|\Pi(k)|$ makes up approximately two thirds of the total power. We take hyperviscous dissipation with $n=8$, and $\nu k_{max}^{2n}=20$, and coefficient $a=0.4$, as recommended in [13].

Figure 3(a) presents compensated spectra $k^{5/3}E(k)$ obtained in simulations with the linear drag. The solid and dashed curves show equilibrium spectra for $\lambda=0.03$ and $\lambda=0.01$ respectively.

Noteworthy is the almost ideal behavior of the compensated spectra: they all show a plateau in the interval adjacent to the forcing scale (the $\lambda=0.01$ spectrum exhibits a small overshoot, as λ is too low). But once again an almost perfect $k^{-5/3}$ spectrum goes along a highly nonuniform flux, especially for large $\lambda=0.03$.

The Kolmogorov constant C_K shown by the thin line of Fig. 3(a) for $\lambda=0.03$, is somewhat lower than the value found in DNS, but EDQNM allows one to adjust it by changing the coefficient a, which we did not pursue.

III. ENERGY PEAK PARAMETRIZATION

Here we shall briefly discuss parametrization of the arrest scale, associated with the spectral energy peak. The dimensional arguments suggest the estimate

$$k_p\sim k_\lambda=(\lambda^3/\varepsilon)^{1/2}, \qquad (2)$$

in terms of the energy dissipation rate ε [or the maximal value of $|\Pi(k)|$ in the energy range]. Similar and related estimates of k_p were proposed by many authors, starting with Lilly [14]. They all assume constant flux, but we saw it vary appreciably over the spectral range $[0,k_f]$. Typically only a small fraction (between 1/5 and 1/3) of ε could reach the energy peak region. The nonuniversal shape of $\Pi(k)$ makes ''universal scaling'' for k_p highly unlikely. In Fig. 4(a) we show the log-log plot of k_p versus $50k_\lambda$ in all our experiments (a factor of 50 was determined by the eyeball). We could derive a factor close to 50, assuming $C_K\varepsilon^{2/3}k^{-5/3}$ spectrum between k_p and k_f. Then k_p is estimated by Ck_λ with coefficient $C=(3C_K)^{3/2}\approx 50–70$ depending on the ill-defined value of C_K near k_f. Figure 4(b) shows the region of the explored parameters. At first glance the data seems to fit the relation $k_p \sim k_\lambda$. Yet, closer inspection reveals that ratio $k_p/50k_\lambda$ varies noticeably, if one of two parameters, λ or ε remains fixed, while the other varies. In a series of experiments presented in Fig. 1 this ratio changed from 0.4 to 1.4. The actual dependence of k_p on λ is thus slower than power 3/2 as shown in [15]. However, limited spectral resolution does not allow us sufficient variation of the linear drag, or energy flux, to draw definite conclusions. Within such limitations, we get an estimate $k_p \approx 50k_\lambda$, for the energy spectra close to $k^{-5/3}$.

We conclude by reiterating our main points: the $k^{-5/3}$ spectrum in the forced 2D turbulence with linear drag seems incompatible with the notion of locality and constant flux. Uniform energy flux coexists with large departures from the 5/3 law, while $k^{-5/3}$ spectra can coexist with highly variable energy flux. Our conclusions are consistent with Maltrud and Vallis [4] that pinpoint highly nonlocal spectral transfers (triads) in the ''inverse cascade.'' Spectral transfers in our DNS experiments look similar to those of [4].

The parametrization of the energy peak as $k_p \approx C(\lambda^3/\varepsilon)^{1/2}$ could be valid within the ''5/3 regime,'' but is not universal. Its possible extension to the variable flux remains open.

While some authors searched for spectral characterization of nonlocality (transfers and fluxes), e.g., the elongated triads of Vallis and Maltrud [4], our results indicate that many essential non-universal features of the inverse cascade could be properly addressed through the physical space analysis, currently in progress [12].

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